Guide to creating a discussion-based math class, with a problem-centered curriculum

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1 Introduction

As a high school student, I had the great fortune to attend Phillips Exeter Academy in Exeter, NH, which uses an innovative problem-centered mathematics curriculum developed by its instructors in the 1980s and continually revised ever since. I absolutely loved it. Unfortunately, when I went to college, all of my mathematics courses were taught in the lecture method, which I found entertaining but not very engaging. When I became a college instructor myself, I decided to write a curriculum in the same problem-centered style for each course that I taught. I have absolutely loved that, too.

This book has two purposes:

- to teach you to write your own problem-centered curriculum, and
- to teach you to teach a discussion-based course using such a curriculum.

I encourage you to use this book as you would any recipe in your kitchen: try following my instructions, and see how it goes. Keep the aspects of it that you like, and in subsequent iterations, modify it to make it your own.

I encourage you to take the opportunity, as you write your curriculum, to make it a reflection of yourself. Emphasize the aspects of the course material that inspire you the most. Include all of your favorite problems. Use examples from your own life. The more that the problems are a conversation between you and your students, the more your course will take on a thriving life of its own.

I also implore you to teach your course in the way that best reflects you and your own individual teaching style. The instructors at Phillips Exeter Academy are so devoted to the idea that each teacher makes the method their own, that when they set out to write a book about how to teach discussion-based classes (the "Harkness method" pioneered and broadly implemented at the school for nearly 100 years), the most honest way they could find to do it was for each instructor to write a personal reflection about their classroom experience [1].

From my own experience, observing others' classes and my own, I can tell you: some teachers have loud classrooms, with the instructor speaking all the time, students interrupting each other and the instructor, people jumping up and sitting down continuously. Other teachers have quiet classrooms, with less talking and with long pauses for thinking. Some teachers have full-class discussions; others have pairs of students at the chalkboard; still others have students working in small groups. In all of these and all of the infinitely many ways one could organize a discussion-based class with a problem-centered curriculum, students learn. Try the methods that I suggest, and modify them to create your own authentic style.

1.1 A pep talk

Hundreds of years ago, before the invention of the printing press, books were rare, and expensive. Hundreds of years ago, before the invention of universal public education, most people could not read. You will find the reflection of this in many of our faith traditions. For example, cathedrals often have beautiful stained-glass windows with colorful pictures illustrating stories from the holy books, which act as picture books for people who cannot read. In many of our faith traditions, part of the worship service involves reading from the holy books. This is because, back when books were rare, and most people could not read, and in fact only certain people were considered holy enough to even read from those books, the one person who could read stood in front of the rest, and read to them from the one available copy of the book. That is a lecture.

All of our students can read, and books are cheap and widely available, so the circumstances of the lecture method's invention have become a historical anachronism.

Lecturing is fun. Preparing lectures is fun. When I prepare a lecture, I read the book, I read my previous lecture notes, I think about everything, I synthesize everything, I figure out why my students might care about this topic, and what's important. I find some examples that clearly illuminate what's going on, and I figure out how to work out those examples as efficiently as possible, for my students.

Writing lectures is fun, but as I do all that work, it dawns on me that I am not the one who needs to learn this subject. I already know this subject. I have taught it before. The person who needs to be finding motivation to study this, and synthesizing information, and working out examples, and figuring out the cleanest, slickest solution is... my students!

A typical lecture goes something like this: motivation, definition, example, example, application. Next day, next topic. Motivation, definition, example, example, application. The big idea in the method of teaching that I will outline in this book is that the *student* will be the one developing that motivation, thinking about those definitions, working out those examples and then figuring out how to apply the ideas to new situations. Instead of the instructor working it out for the students while they watch, each student will work it out, and because they will get stuck, they will come to class.

I have given some great lectures over the years, lectures where I brilliantly motivated why students should care, gave compelling examples that got right to the heart of the topic, presented it in such a way that students were on the edge of their seats, wondering what would come next – will it work, will it work? – And then it *did* work! And it was practically a miracle! And the students were excited, and I was excited, and I was a hero, and everyone went home happy. It's nice to be the star. But then, on the exam, some of the students couldn't solve the problems on the topic I taught them that day. Why not? I had taught them that; I taught them very well, with several well-chosen examples, from easy to hard, and the students were paying attention, they were so excited – so how did they not learn?

Studies show that retention of understanding from the lecture method is about 5%. Many people feel like they learn more than 5% of the content, but a lecture can lull you into a false sense of understanding. During the lecture, and after the lecture, you're happy and you feel like you understand all of it, but then when you sit down to do the homework problems, they often seem harder than the lecture. No kidding! Because the instructor was the one explaining things and working out examples, and the student was just sitting there, passively taking it in.

Let me tell you a story about learning. One day one of my students, Morgan, came to class, and there was a homework problem that she had not been able to solve. No one in her group had solved it either, so after they discussed all of the other problems, they came back to this one. They struggled with it together for a while, and then they called me over. "Professor," Morgan said, "we can't solve this problem." "You *can* solve it," I said, "I know you can, you just haven't sold it yet." "No, we can't. Professor," she said – and this is why I love this story, because she articulated the issue so precisely, so assertively – "Professor, *I don't have the tools* to solve this problem." And I said to her, "you do have the tools. I promise you that you do."

And a promise is not something that I take lightly, but still she did not believe me, so I told the group, write, at the top of the board, what you are starting with, and write at the bottom of the board where you want to end up – this was a problem about proving some kind of formula – and see if you can get from one to the other. I told Morgan and the rest of her group, just start with the line at the top, and write the next line, see what you get, see what happens. And then I walked away.

And then, it took time. Maybe five minutes. Five minutes is a long time for a group of students to think together and work together on something that they have already thought hard about before. But after about five minutes, from across the room I heard this sound, the most wonderful sound – "Oh!"

"Oh," she shouted, and it was Morgan, Morgan who had had this insight, Morgan who had not believed that she had the tools, had an idea, right there in class, and she used that idea, and she solved that problem. That's *learning.* That kind of learning won't happen for every homework problem, or every day, or maybe even every week. But when it happens, students learn. This style of curriculum and classroom organization is designed to create opportunities for students to have that kind of experience. It is designed to create opportunities for students to *learn.*

Here's how it works. For a typical lecture class, the plan might be: day one, topic one, motivation definition example example application. Day two, topic two, motivation definition example example application. And so on. For this curriculum, instead of the instructor demonstrating these things for the students, the students work them out, and will come to class with their solutions and questions. That's it! That's the game.

But that's not quite all. That's a little fast, isn't it? On day one we introduce the first topic, do everything with it, and then turn the page. On day two, we introduce the second topic, do everything with it, and then turn the page. That's too fast. So here's the actual setup of this curriculum: For given topic, one day the homework problems will explore it a little bit, and then students will come to class and talk about it. The next day, the problems will explore it some more, so that students really try to figure out what's going on, and come in to class and talk about it. The third day, we'll really get into it, get to the heart of the thing, write down some definitions, and make everything precise. Then for the next couple of days after that, there will be problems with examples, so that students practice the things they have learned, and then a few days later extend it to some applications. This way, students have time – time to reflect, to get confused, to figure things out, to have some ideas, and to learn.

For a given homework assignment, there will be problems on many different topics. For one topic, maybe we're just beginning to explore it. For another topic, we've already been exploring it for a few days, so now is the day when we get to the heart of things and make them precise. And for some other topic, we've been studying it for a while, so today we'll work out some examples and apply it to new situations. That's it! That's the philosophy.

Let me tell you another story. A student came up to me at the end of a class period, quite upset, and said, "my group discussed the homework problems, and I understand partial derivatives now, and directional derivatives – but I really don't understand the gradient at all! I asked questions and my group tried to answer them, but I still don't get it!" And to that I say – "yes! We're doing it right!" not because I'm a sadist, but because, get this, that student understands *at least* half of the material. So we were beating 5% retention of understanding by a factor of 10. Plus, in no way was this student lulled into a false sense of understanding. He knew what he doesn't understand, and he was highly motivated to figure it out. And sure, he doesn't understand the gradient today, but as explained above, we'll hit it again on tonight's homework, and we'll hit it again on the next one, and by next week, he'll understand the gradient.

Learning is not something that someone can teach a student who is sitting quietly and listening to them. Learning is not something that someone can a student who is copying down their solution from the board. Learning is something that each person does for themself. It may not happen for every student every day, or even every week, but when it does, it's beautiful.

This method of instruction works. A recent meta-study of 228 studies comparing inquiry-based teaching methods to the lecture method overwhelmingly showed that inquiry-based learning is better. "So much so," a recent article about it says, "that the authors found it ethically questionable to make students attend lecture-based courses, given all that we know about how ineffective they are. . . If the studies had been medical experiments, they probably would have been" stopped in the middle, "because the treatment being tested was clearly more beneficial." CITATIONS

This method of instruction respects our students as human beings. Lecturing at them does not recognize their humanity. I feel that it is important to speak with students in the classroom the same way we speak with them outside of the classroom, and the same way we speak with everyone else in our lives, to respect them, and recognize their humanity. I feel that it is important to respect their time, not wasting it on lectures that they could watch at home, and by making good use of every minute that we have together in the classroom with thoughtfully structured classroom culture and group work, and making good use of every minute they spend on homework by crafting good problems that give them opportunities to truly learn.

In the words of Marianne Williamson, "our deepest fear is not that we are inadequate. Our deepest fear is that we are powerful beyond measure." I believe that our students are capable of so much, perhaps even more than they believe in themselves. With the curriculum method and the class structure described in this book, students will find a classroom where their instructor respects them, challenges them, and never, ever underestimates them.

2 History

Here is a brief overview of active learning mathematics classrooms, and the basis of this pedagogy.

Socrates (470–399 BC) used his eponymous *Socratic method* for teaching his students. In this method, the teacher asks a series of questions that is designed so that the student will discover something. In the Socratic method, the teacher is expecting a particular answer from the student; the question is designed as a "fill in the blank." Some of the Socratic dialogues are about mathematics; for example, one is about how to create a square whose area is twice that of the original square.

R.L. Moore (1882–1974) used his eponymous *Moore method* at the University of Texas from 1916–1969 [5]. His method is so well known in college mathematics that people often assume that any "inquiry-based learning" in a college math course is the Moore method. Each student is given a list of problems of increasing sophistication, which takes the student from wherever they start, to mastery of the course material. Class time consists of individual student presentations of solutions to these problems. The Moore method prohibits students from using textbooks, uses only very brief lectures in class, and prohibits communication or collaboration between classmates. "That student is taught the best who is told the least," said Moore. The educational philosophy is based on three student-centered goals [5]:

1. To independently develop a solution and supporting argument,
2. to communicate that solution via a supporting argument, and
3. to defend that argument.

Moore himself has fallen out of favor due to his well-documented racism, and his reported sexism and anti-Semitism. Still, his pedagogical influence lives on in the college mathematics teaching community, as many of his former students continue to use his method and actively work to teach others how to do it. For a resource similar to this one but for the Moore method, see [2].

Phillips Exeter Academy's mathematics instructors (1980s-present) wrote the problem-based mathematics curricula on which the methods described here are based. PEA is a private high school, where each class has approximately 12 students with one instructor seated around an oval table. All classes are conducted in the discussion-based *Harkness method*, which the school developed in the 1930s. Initially, PEA math courses used standard math textbooks, but after about 50 years, the instructors decided to write their own curriculum that made better use of the discussion-based format.

The books are each written for a year-long mathematics course with an integrated study of many topics. The first one the instructors wrote was *Math 2* (geometry), followed by *Math 3-4* (trigonometry and pre-calculus), *Math 4-5* (calculus), then *Math 1* (algebra), *Math 6* (multivariable calculus), and *Discrete Mathematics*, all of which are freely available at [6]. They have been in constant use since the late 1980s, and are revised by a committee of faculty members each summer.

I (Diana Davis) learned all of my high school mathematics in 1999–2003 through Phillips Exeter Academy's problem-centered, discussion-based "Harkness method" classes. After majoring in mathematics in college, I returned to PEA as a one-year intern in the mathematics department for the 2007–2008 school year, teaching four classes over the course of the year, coaching four sports, and living in the dormitory. It was a huge transition for me to go from being a successful student under this method to being a successful teacher, as the skills and actions required are almost opposite.

In graduate school and during the first two years of my postdoc, I taught lecture-based classes, because that was the expectation in the mathematics departments. Lecturing was fun, but I felt that it was a monumental waste of time for me and for my students (for more, see §1.1). In the summer before the third and final year of my postdoc, I read a book of reflections by PEA faculty members [1], and made the decision to switch all of my classes to a problem-centered, discussion-based method, to the greatest extent possible. I was able to teach one class in that method that year, and the data I collected showed that my students did the same on exams and better on other skills than students in the parallel lecture-based sections [3]. Since then, I have taught every possible in this method, adapting a PEA problem book [6] or writing my own as needed. As of this writing, I have taught college courses on discrete mathematics, calculus I, multivariable calculus, introduction to proof, real analysis, and a senior seminar on billiards in this method.

3 Creating a problem-based curriculum

3.1 How to write a problem-centered curriculum in five steps

1. **Understand everything:** Put the entire curriculum in your head. To do this, you might teach the class all the way through as a lecture, or do all of the homework problems that are traditionally assigned for it. If you are designing the course from scratch, ensure that you have a complete understanding of everything that will be in it and how it all fits together.

2. **Know the course goals:** Make a list of the ideas you want your students to understand, the skills you want them to acquire, and the problems you want them to be able to solve by the end of the course.

3. **Create a thematic map of topics:** Determine which topics your course will study, and how they are connected. Put basic ideas at the bottom, and the course goals from the previous step at the top. Create a flow chart, mapping which ideas depend on which others, to help you keep track of the flow of interconnected ideas. See §3.2.

4. **Write the problems:** For each of these things, write a thread of problems that builds this ability. Whenever possible, start with an exploratory problem that induces a sense of wonder, an innate questioning. Then write a series of problems building skills bit by bit, through student discovery. Culminate the series with problems that will truly challenge them. Create follow-up problems that recall or extend these skills. See §3.3–§3.9 for more on creating problem threads and writing individual problems.

5. **Create the book:** For each thread of problems, allocate them with zero, one or two per day, so that the skills are built over the course of many days. Leave enough time between problems on a given topic that students can discuss each problem in class *before* they must do another problem that builds on that problem. See §3.10–§3.12 for more on ordering problems.

Action Task 3.1. Make a list of all of the ideas and skills you want your students to understand and master by the end of the course. A good place to start is the table of contents for a textbook associated to the course; the section titles tend to list most of the topics covered. You will probably want to add things to that list.

3.2 Create a thematic map

In mathematics, many ideas are interrelated, so I suggest creating a thematic map. For example, the content of a multivariable calculus course can be broken into derivatives, integrals, and vector calculus. Each of these can be developed largely independently (and thus simultaneously), but there are some interdependencies.

I created the thematic maps shown below to help me organize courses on multivariable calculus (Figure 1) and discrete mathematics (Figure 2). The idea of the map is that the goals of the course are at the top:

- By the end of multivariable calculus, students need to be able to apply the big theorems (Green's Theorem, Stokes's Theorem, Gauss's Theorem), and find extrema of functions in general and using Lagrange multipliers.

- By the end of discrete mathematics, students should understand important applications of graph theory (the 4-Color Theorem, the Traveling Salesman problem), what outlier analysis for gerrymandering is measuring, various election methods, and the Huntington-Hill apportionment method used to apportion representatives to each state.

We start with the basic ideas at the bottom of the map, and then everything builds from those towards the course goals.

An arrow goes from A to B if you need to do topic A before doing topic B. In each course, I had several threads being developed simultaneously, in each assignment. The three main threads of multivariable calculus:

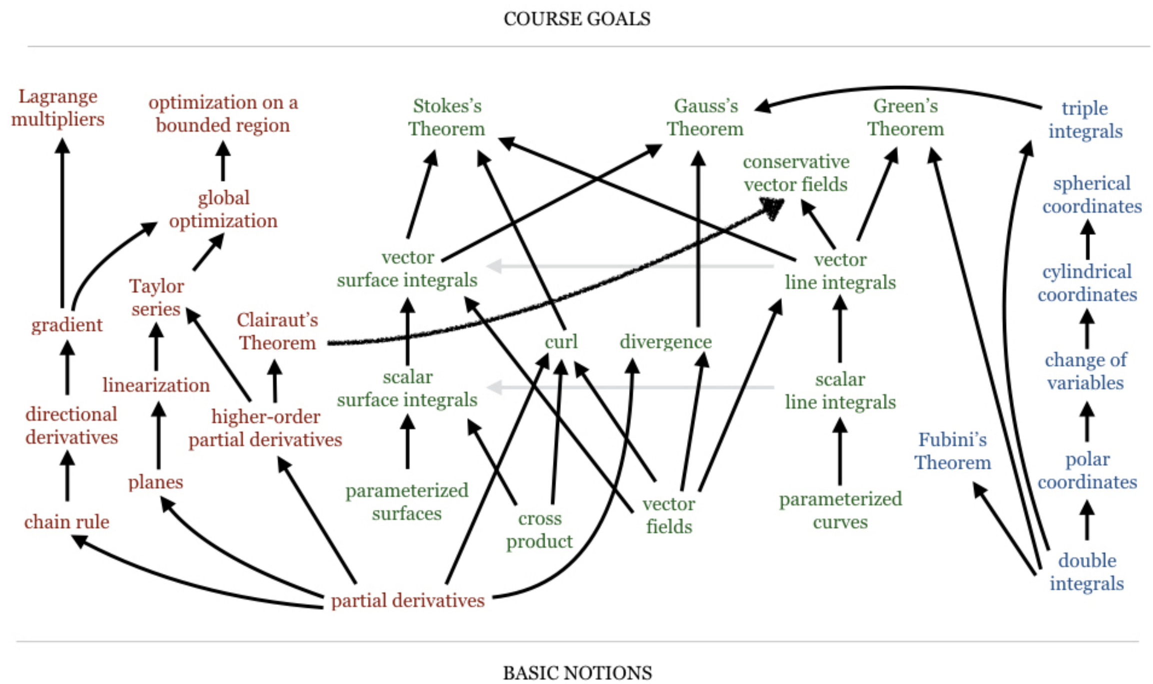


Figure 1: A thematic map of multivariable calculus, showing topic dependencies. Derivatives are in red on the left, integrals are in blue on the right, and vector calculus is in green in the center. The chalk arrow was added, and the grey arrows were deleted, after teaching the course (see text).

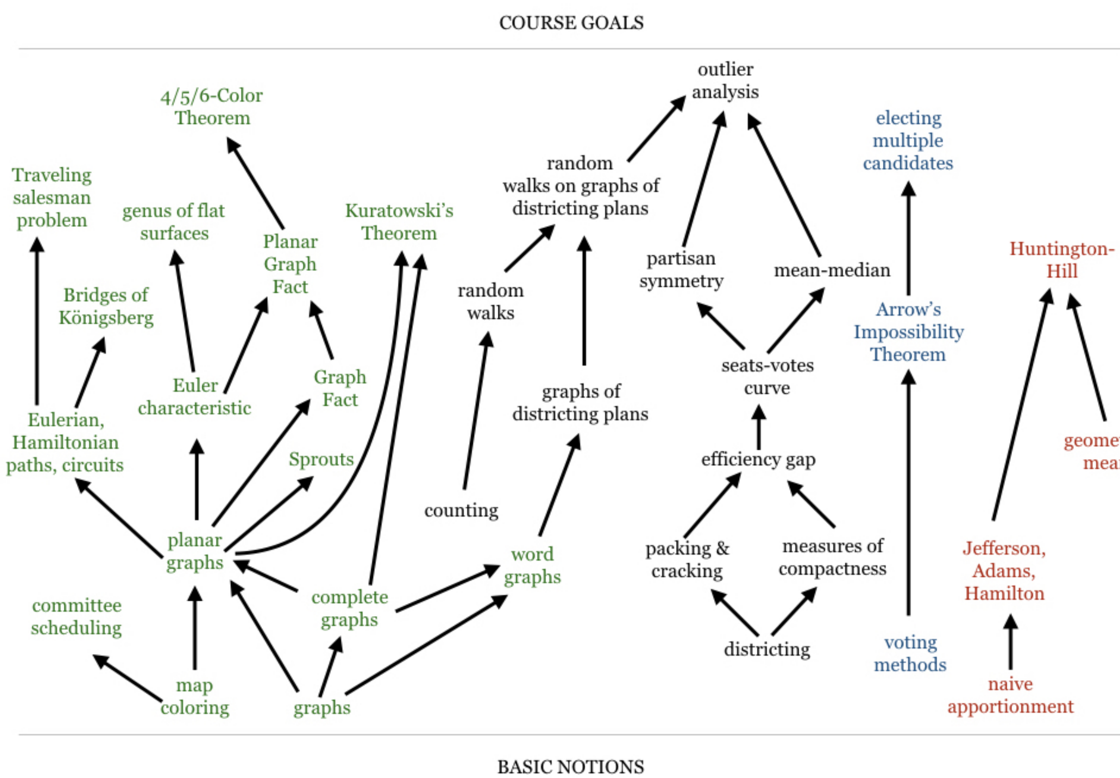


Figure 2: A thematic map of discrete mathematics with gerrymandering. From left to right, graph theory is in green, gerrymandering is in black, voting theory is in blue, and apportionment is in red.

- *Derivatives:* partial and directional derivatives, gradient, extrema, Lagrange multipliers
- *Integrals:* double integration, new coordinate systems, triple integration
- *Vector calculus:* parameterized curves and surfaces, scalar and vector line and surface integrals, Green's Theorem, Stokes's Theorem, Gauss's Theorem

  Plus several other topics that are related but don't directly build towards the course goals:

- *Approximation:* tangent planes, linearization, Taylor series
- *Other coordinate systems:* polar, cylindrical, and spherical coordinates; change of variables

With the map in hand, I took the sequence of problems that I constructed for each thread, and then made sure that where threads were interconnected, I had developed everything by the time I needed it. This map also helped me improve things for future iterations of the course:

- I thought that Clairaut's Theorem could be thrown in anytime, but it turns out it is needed to prove something about conservative vector fields. Thus, while teaching the course, I added in an arrow between these topics, so that I would include Clairaut's Theorem earlier next time.
- I had always taught scalar and vector line integrals before scalar and vector surface integrals, thinking that it was better in both cases to study the one-dimensional thing before the two-dimensional thing. Thus, I had two arrows in the chart pointing from line integrals to surface integrals. However, I found that the students solidly understood line integrals, but had trouble understanding surface integrals. So I decided to remove these arrows and develop the two simultaneously the next time.

Action Task 3.2. Write each of your topics from Action Task 3.1 on its own post-it note. Arrange the topics in a thematic map of dependencies, as described above. Consider doing this on a whiteboard, so that you can draw in arrows.

3.3 Core guiding principles of a problem-based curriculum

- Students discover the material through doing problems, with no lectures or reading.
- These problems are designed to build understanding of the material, with each problem building on the understanding gained in the previous problem on that topic.
- A given homework assignment contains problems on many different topics, so that understanding of many topics is built simultaneously, and understanding of a given topic is built over many days and weeks.
- Notation and terminology *follow* student understanding of the notion that the terminology expresses.
- Formulas and theorems *follow* students computing examples and noticing patterns.
- Students discover the mathematics just as mathematicians have discovered them over the past centuries, except that we guide them with hints, since the ideas are often subtle and brilliant.

3.4 Some tips for writing problem threads

When constructing the thread, just sketch out the idea of each problem. For each topic or skill, the goal is to create a sequence of problems building it step by step. I like to write down just the purpose of the problem, such as "estimate vector line integral" or "accumulation function for $y = 2x$" while I am constructing the problem threads. Then I go through later and write the actual problems. This helps with doing only one kind of thinking at a time – first global thinking about what kind of problems you need, and later specific choices of examples and wording – and saves time in case you change your mind later about the kind of problems you need.

If you have lecture notes for a course: When I start with a course for which I have lecture notes available, I simply turn each lecture into a problem thread.

- For each definition: I write a problem that includes the definition and asks the student to apply it, often with an example and a non-example.
- For each example: I simply turn it into a problem asking the student to work out the example.
- For each lemma, proposition or theorem: I write down the statement and either ask the student to apply it (if the proof is out of reach), or write down the steps of the proof and ask the student to justify why each one is true (if this is within reach), or ask the student to prove the result (if they have all the tools from previous work).

In order to put some of these things within students' reach, it is sometimes necessary to create priming problems (see §3.8) that were not in any lecture, which will precede by a day or two the problem in which they must use that idea.

For each topic, either do it or don't do it. If you choose to include a topic in your course, do it right: Write a problem thread that introduces it, explores it, defines it, and allows students to gain a full understanding of it. If you are not willing to give a topic this much space, omit it from the course. Resist the temptation to cram the beginning, middle and end of a topic into a single problem and expect students to get the entire point in one sitting.

Action Task 3.3. Find an existing problem-centered curriculum, such as one from Phillips Exeter Academy [6], on a topic that you know very well. Choose one topic, such as "factoring polynomials" or "finding the circumcenter of a triangle" or "the derivative of e^x," and circle all of the problems in the text that build that skill or idea. Then think carefully about the thread of problems:

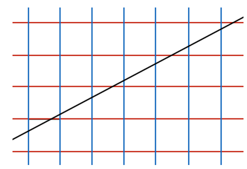
- What do you think are the problem-writer's goals for what the student should be able to do or understand at the end of this thread?
- What basic skills or understanding does a student need, before they can start this thread?
- How did the problem-writer motivate study of this topic? Identify the first examples or ideas that were introduced, and think about why the writer made this choice.
- When was formalism (a precise definition, or statement of a result) introduced, and how?
- If you were using this thread for your students, how would you modify it?

3.5 Example of a thread of problems: finding the period of a billiard trajectory on the square with slope p/q

This class was a senior-level college seminar on mathematical billiards, which is my research area. One of the beautiful results in this area is that, if you shoot a billiard ball in a square table with a rational slope, the trajectory looks like a Celtic knot, and also we can say how many times it bounces before it repeats. This series of problems begins with two exploratory problems that encourage students to wonder and conjecture about periodic paths. Then there is a series of problems that first asks them to sketch periodic paths, then gives them the tools to do so precisely, and then asks them to sketch more complicated periodic paths, all the while building a family of examples, and an intuition about the system. Over the course of doing this, they will have conjectured a relationship between the slope p/q of the trajectory, and its period. The last question in the series asks the student, finally, to prove this relationship.

This class met Monday, Wednesday and Friday for 50 minutes. The students worked on page 1 on the first day of class. Page 2 was their homework, which we discussed in the second class, and page 3 was that night's homework, discussed on the third day of class, and so on. The page number thus indicates the class number, and the problem number the order within that assignment.

Page 1 # 1. Draw a line on an infinite square grid, and record each time the line crosses a horizontal or vertical edge. We will assume that the direction of travel along a line is always left to right. We could record the line to the right with the sequence ... ●●●●●●●●●●●●●● ..., or we could assign A to horizontal and B to vertical edges, and record it as $\ldots BABBABBABBA\ldots$.



(a) What is the slope of the line in the picture?

(b) Record this *cutting sequence* of colors, or of As and Bs, for several different lines. Describe any patterns you notice. What can you predict about the cutting sequence, from the line?

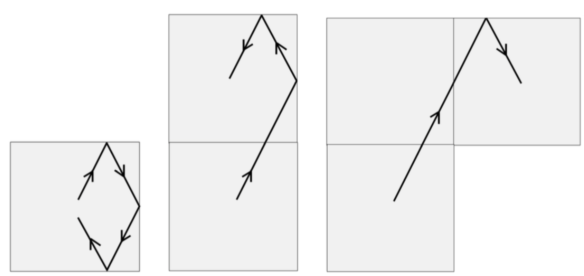
(c) What should you do if the line hits a vertex?

Page 1 # 2. Consider a ball bouncing around inside a square billiard table. We'll assume that the table has no "pockets" (it's a billiard table, not a pool table!), that the ball is just a point, and that when it hits a wall, it reflects off and the angle of incidence equals the angle of reflection, as in real life. (We'll prove this *billiard reflection law* later.)

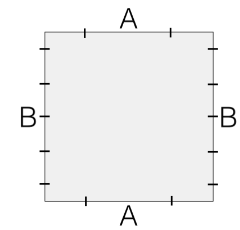
(a) A billiard path is called *periodic* if it repeats, and the *period* is the number of bounces before repeating. Construct a periodic billiard path of period 2.

(b) For which other periods can you construct periodic paths?

Page 2 # 5. A powerful tool for understanding inner billiards is *unfolding* a trajectory into a straight line, by creating a new copy of the billiard table each time the ball hits an edge. Two steps of the unfolding process are shown for a small piece of trajectory. Draw several more steps. Then use this unfolding to prove that any trajectory with slope 2 yields a periodic billiard trajectory on the square. (We always assume that one edge is horizontal.) Which other slopes yield a periodic billiard trajectory?



Page 3 # 2. Show that the cutting sequence corresponding to a line of slope $1/2$ on the square grid is periodic. Which other slopes yield periodic cutting sequences? What can you say about the period, from the slope?



Page 4 # 3. In Page 3 # 4, we ended up with a trajectory of slope 2 on the *square torus* surface. The picture to the right shows some scratchwork for drawing a trajectory of slope 2/5 (or −2/5) on the square torus. Starting at the top-left corner, connect the top mark on the left edge to the left-most mark on the top edge with a line segment. Then connect the other six pairs, down to the bottom-right corner.

(a) Explain why, on the torus surface, these line segments connect up to form a continuous trajectory. Find the cutting sequence corresponding to this trajectory.

(b) Exactly where on the edges should you place the marks so that all of the segments have the same slope?

Page 5 # 1. Draw an accurate picture of a trajectory on the square torus with slope 3/4, and do the same for two other slopes of your choice. For each one, find the corresponding cutting sequence. *Hint*: Use graph paper. You can print out any size you want, from the Internet.

Page 5 # 2. We saw that a billiard trajectory on the square table can be *unfolded* to a line on the square torus. Going the other way, a line on the square torus can be *folded* to a billiard trajectory on the square table. Choose one of your trajectories from the previous problem. On a loose piece of paper (or a transparency), draw a large accurate picture on a square, and fold it in half twice, like a paper napkin. Hold it up to the light, and trace through to see the torus trajectory, transformed into a billiard on the square! Find the corresponding cutting sequences on the square torus, and on the square table. What are their periods?

Page 7 # 2. You have seen that studying a line on the square grid is essentially the same as studying a straight trajectory on the square torus. Explain how to go from one to the other, in both directions.

Page 8 # 2. Construct a periodic billiard path on the square table with an odd period, or show that it is not possible to do so.

Page 10 # 2 . How many billiard paths of period 10 are there on the square billiard table? Of period 14? Construct (make a mathematically accurate sketch of) each of these.

Page 11 # 2. Prove that a trajectory with slope p/q (in lowest terms) on the square billiard table has period $2(p+q)$. *Note:* We have stated this in class several times, with several different arguments; the goal here is to write down a clear, rigorous proof.

By the time I assigned the proof in Page 11 # 2 in the fourth week of class, the students had conjectured the result many times. In fact, they were stating it in class as though it was an obvious fact, and over the previous week, various students had come up with at least three different methods for proving this result. The theorem was in the air, and all of the students had the necessary tools. I wanted them to prove it, but first, I wanted them to really, really *want* to prove it. The parable in §3.9 gives a more colorful illustration of the same idea.

Action Task 3.4. Choose one (or more) of your topics from Action Task 3.1, and write a thread of problems that builds student skills and understanding of that topic. Of course, threads may be interconnected, as you likely saw in doing Action Task 3.2.

3.6 Some tips for writing problems

- *Break problems into parts.* I have found that, when a problem asks students to do several things within a single sentence, many students somehow forget to do some of the instructions, and then aren't prepared to think about those aspects of the problem during class. I get around this by explicitly breaking the problem into parts (a), (b), etc., one for each instruction. This also helps if you assign e.g. 8 problems for homework and have 13 students writing solutions on the board, so that you can write each problem part as a separate problem number, and every student has something to write up independently.

- *Include lots of pictures.* Problems come alive with figures to illustrate what is going on. Ideally the student would be able to visualize or draw the figure on their own, but giving them the figure can help them get to the point of being able to think about the heart of the problem. You can also carefully choose what kind of pictures to draw, or what angle to show them from, to prime students to have certain ideas (see §3.8).

- *Create old friends.* Phillips Exeter Academy's curriculum has an "Alex in the desert" problem that shows up in every level of the curriculum, and is beloved by students. In my own multivariable calculus materials, I use the exact same surface (an upside-down paraboloid) over and over, to introduce every idea about derivatives: students take partial derivatives and directional derivatives, compute tangent vectors, tangent lines and tangent planes, all at the same point on that same familiar surface. Similarly, I use the same topographical hiking map for all of our level curve and gradient problems. It's easier to understand a new concept when everything else about the problem is familiar.

- *Infuse your personality into the problems.* A classroom is essentially a relationship, between a student and the instructor, and between students. Let your problems be a conversation between you and each student. Talk to them in the problems. Tell them about your perspective on the ideas. Infuse your passions (hiking? farming?) into the problem content. I have noticed that the first time I constructed a curriculum, when I took most of my problems directly from Phillips Exeter Academy's materials and kept the same writing style for those I added, my students referred to the problem-writer as "they," as in "I think what they want us to do here is..." Now that I have substantially edited the Exeter problems and added a large number that are unmistakably my own, my students consistently refer to the problem writer as "she."

Action Task 3.5. Go through the problems you wrote in Action Task 3.4.

- If any of your problems have a lot of instructions, rewrite them with multiple parts, and see if you like it better.

- If any of your problems ask students to draw a complicated or new type of picture, consider providing one for them instead.

- If you took problems from another source, try rewriting them from scratch in your own style.

3.7 Scaffolding

Mathematics has been created by highly educated, very intelligent people, over hundreds and thousands of years, after many failed attempts. If students are going to discover these same ideas while in high school or college over the course of just a single semester, they will need help. The goal of a problem-centered curriculum is to build the material through doing problems, which we must carefully design so that students can solve them, with each problem requiring slightly more of the student than the previous one.

The entire curriculum is essentially scaffolding. For example, in order for students to be able to understand Green's Theorem, they need to understand vector line integrals and double integrals (see Figure **??**). To understand vector line integrals, they need to understand vector fields and scalar line integrals. To understand scalar line integrals, they need to be able to parameterize curves.

The entire course works like this: we identify the course goals, and then help students build all the tools they will need in order to understand those ideas, starting from the basics.

For this style of curriculum, students need to have an idea in order to solve each problem. Some problems are hard. We can break them into parts.

> *Phillips Exeter Academy, Mathematics 1, problem 327.* Twelve flags are evenly spaced around a running track. Ryan started running at the first flag and took 30 seconds to reach the sixth flag. How many seconds did it take Ryan, running at a constant rate, to reach **(a)** the 10^{th} flag for the first time? **(b)** the 8^{th} flag for the 2^{nd} time? **(c)** the n^{th} flag for the m^{th} time?

If the question were simply part **(c)**, without **(a)** and **(b)**, this problem would fail many students. Where to begin? They might write down an answer like $5mn$, and when they came to class they would be completely unprepared to receive an explanation of the correct solution. The concreteness of parts **(a)** and **(b)** allows students to draw a picture and explicitly count things.

Part **(a)** encourages students to notice that there are 5 (not 6) "gaps" from the first flag to the sixth flag, and that there are consequently 6 seconds of running (not 5) between flags. Part **(b)** encourages students to notice that there is one full lap (not two) before reaching a given flag for the second time. All of this scaffolds students to be able to come up with all the parts of the rather complex answer $6\big(12(m-1)+(n-1)\big)$. The first two parts also give concrete numbers that students can use to check whether their general solution in **(c)** is correct.

Action Task 3.6. Go through the problems you wrote in Action Task 3.4. If any of your problems might be too hard for most of the students in your course, consider breaking them into parts, either as multiple parts of a single problem, or as multiple problems that will be done on different assignments and discussed in between.

3.8 Priming

As discussed in §3.7, the goal of the curriculum is for students to build the material through doing problems. Some of the problems may require the student to come up with some particular good idea. We can make it easier for them to have those ideas, without taking away their agency, by *priming*: making them think about a related idea beforehand, so that they are more prepared to spontaneously make the connection.

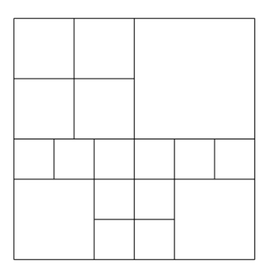
Here is an example. One day, students do the following problem:

1. In the game of *football*, there are two ways to score points: a *touchdown*, worth 7 points, and a *field goal*, worth 3 points. What are all of the possible scores that a football team can have?

Students begin by writing down possible scores: 0, 3, 6, 7, 9, 10, 12, 13, 14, 15, . . . eventually someone wonders if *every* number 12 and higher is possible. After 15 or 20 minutes of thinking in a group, someone usually has the following idea: if there are three numbers in a row that are possible scores, then by adding a field goal to each of them, you can get the next three numbers, and you can repeat this forever. Since we can get 12, 13 and 14, every number above 12 is possible.

A few days later, students do the following problem:

2. (Phillips Exeter Academy, Mathematics 2, problem 84) It is a simple matter to divide a square into four smaller squares, and – as the figure at right shows – it is also possible to divide a square into seventeen smaller squares. In addition to four and seventeen, what numbers of smaller squares are possible?



There is no obvious connection whatsoever between this problem and the football problem. Yet at some point, a student will have the idea that anytime you have a square, you can break it into four smaller squares, thereby increasing your number of squares by 3.

Thus as soon as you have three numbers in a row that are possible, you have all higher numbers as well, just as in the football problem.

There are many ways to prime students with ideas. In my multivariable calculus class, I give my students several examples of vector fields, have them compute the *curl* vector of each one, and then ask them to say what they think the *meaning* of the curl vector is. This is a pretty hard thing to guess from scratch. So, earlier on the same page, I have given them a (seemingly unrelated) problem about the Right Hand Rule, with lots of examples and pictures. With the Right Hand Rule staring them in the face, many students notice that the curl vector seems to represent something like the Right Hand Rule for the vector field, which is essentially what it is.

Similarly, notice that in the illustration for the squares problem above, one square has been broken into four smaller squares. Staring at the picture helps the student come to that idea on their own.

Action Task 3.7. Go through the problems you wrote in Action Task 3.4. If any of your problems require the student to have one particular idea, consider writing a priming problem to go earlier.

3.9 A parable

Legend has it that one day Socrates and Plato were walking down the beach, and Plato expressed his fervent desire to become as wise as Socrates. Socrates didn't answer him, but instead said, "Walk with me into the ocean." So they turned and walked into the sea together.

The water started out around their ankles, then rose up to their knees. When the water was shoulder height, Socrates asked Plato, "What is it exactly that you want from me?" "Wisdom," Plato answered. Socrates grabbed Plato's head and pushed him down under the water. After a half a minute or so Socrates let Plato up and asked him again, "What is it that you want?" "Wisdom" was again Plato's answer. Socrates shoved him back down under the water.

After a time, Plato began to run out of air, and he struggled to get his head above the surface. He punched and kicked and flailed to get free, but Socrates held him down. At the very last moment, Socrates let him up and asked again that simple question, "What is it that you want?" Plato gasped oxygen into his lungs and shouted, "Air! I need air!" Socrates told him calmly, "When you desire wisdom as much as you desired a breath of air, then you shall have it."

When you write your curriculum, and when you teach your course, give your students things – notation, terminology, a formula, a theorem – only after they really, really want them.

Action Task 3.8. Go through the problems you wrote in Action Task 3.4. If any of your problems give students a result, notation, etc. before they really understand that idea, consider re-ordering the problems to put exploration and examples first, and formalism after.

3.10 A stair-shaped curriculum: homework day by day

Consider a curriculum where you want to teach a number of different ideas or skills, through problems that gradually build up students' understanding. The following table shows one way to do it. The idea is that students are working on several different topics at once, and in a given homework assignment, their level of mastery of one topic will be ahead of their level of mastery of another topic that they are exploring on the same day.

In this example, the topics are being introduced in the order:

$\cdots$ limits $\to$ derivatives $\to$ differentiability $\to$ partial derivatives $\to$ gradient $\to$ extrema $\cdots$

	$\cdots$	Day 5	Day 6	Day 7	$\cdots$
application	$\cdots$	limits	derivatives	differentiability	$\cdots$
examples	$\cdots$	derivatives	differentiability	partial derivatives	$\cdots$
definition	$\cdots$	differentiability	partial derivatives	gradient	$\cdots$
motivation	$\cdots$	partial derivatives	gradient	extrema	$\cdots$

The idea here is that, for example, on Day 6, the students are at a high level of understanding of derivatives, they are practicing their understanding of differentiability, they are just at the culmination of partial derivatives, and they are just exploring for the first time the ideas of gradient.

Of course, you will want to take more than three days to explore each topic, and you will likely want to work on more than four topics per day.

You may wish to have two problems on a given topic on the same day, but make sure to space them out enough that students can come in to class and get help with any problems they couldn't solve, before having to tackle the next one, and also to give them enough time for the ideas to sink in over many days. If you have two problems that build different skills or ideas towards a certain topic, it is fine to put them both on the same day, since one problem does not build on the other.

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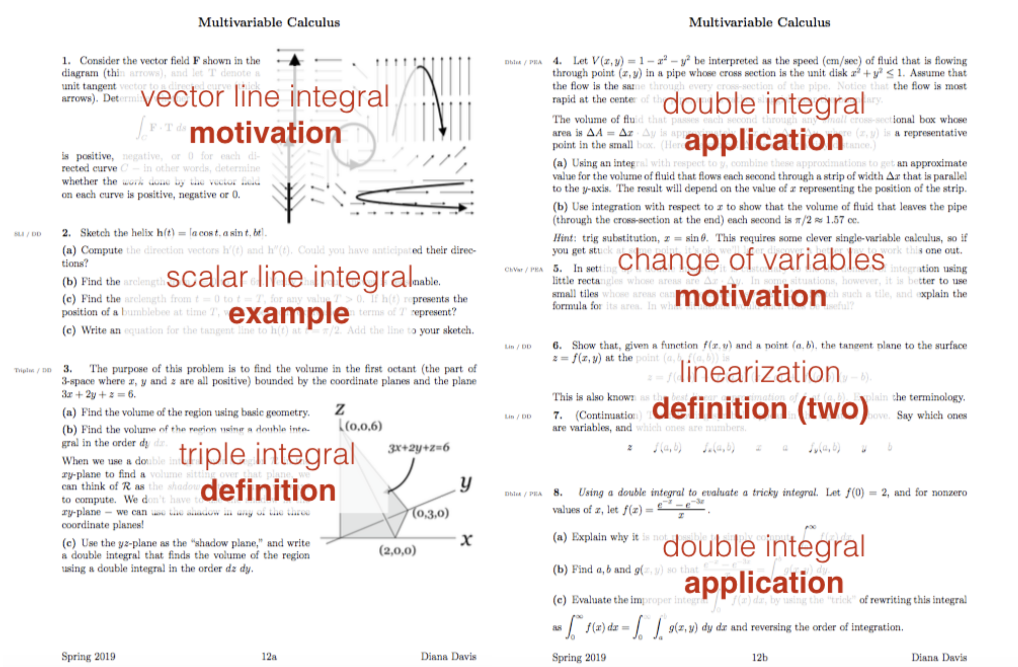


Figure 3: One night's homework: 8 problems on 6 topics, with different levels of understanding

Action Task 3.9. Find an existing problem-centered curriculum, such as one from Phillips Exeter Academy [6], on a topic that you know very well. Open to a page in the middle of the book, and think about the problems that appear there:

- For each problem, identify the skills or ideas that it is building.
- For each problem, try to guess where it is in the thread of problems on that topic, using the categories motivation / example / definition / application from above, or your own.

3.11 Decide the topic ordering in the book

The hardest part for me is to choose which new topics to introduce on which days. The first step is to make a list of all of the topics in the course, which includes everything from the thematic map, plus additional things that need to go in but aren't big enough to be themes. For the multivariable calculus example, this includes things like changing the order of integration, computing spherical and cylindrical integrals, and closing off a region to apply Green's Theorem. Then the job is to decide which days to introduce which topics.

Note that the flow chart of topic dependencies (§3.2) still leaves a great deal of freedom in topic ordering. In particular, the "standard lecture order" from the table of contents of a textbook for the course surely respects the topic dependencies. When course topics are arranged by chapter and section, they tend to start at the "root" (basic ideas) of the tree of related topics, go all the way to the ends of the "branches" (course goals), before starting on a whole new new topic. For example, a standard course in multivariable calculus does derivatives, then integrals, then vector calculus (refer to Figure 1).

When I reordered the course topics, my philosophy was as follows:

- My goal was to get to the hardest topics (Green's, Stokes's, and Gauss's Theorems) as early as possible, to give students as much time as possible to absorb the hardest ideas.
- The newest ideas for the students would be vector fields and double and triple integrals, so I started those at the very beginning, on days 1 and 2 (see Figure 5).
- Any topics that did not directly build towards those (limits, optimization), I filled in along the way, wherever there was extra time.
- Ideas (vector fields, curl, divergence) are introduced *well before* the results (Green's, Stokes's, and Gauss's Theorems) that will need them, to give students time to understand the ideas.
- Tools (cross product, polar coordinates) are introduced *just before* the results (surface integrals, double integrals in polar coordinates, limits) that will need them, so that their purpose is clear.

Figure 4 shows the result, which is that the topics ended up ordered very differently in my curriculum than in the lecture order.

To make the day-by-day calendar, I start from the end of the course, determining which days I want to introduce the big goals. For the multivariable calculus example, the big goals are optimization, Lagrange multipliers, Green's Theorem, Stokes's Theorem, and Gauss's Theorem. For a 28-day semester, I wanted to do Gauss's Theorem on the third-to-last day, Stokes's Theorem three days before that, Green's Theorem about 2/3 of the way through the semester, optimization also about 2/3 of the way through the semester, and Lagrange multipliers about six days before the end. I placed these labels on my course calendar and filled in everything else afterwards. Figure 5 shows the result.

Note that this day-by-day schedule includes more than the topic list in Figure 4 because I often broke a topic down into smaller pieces and said which day I would introduce which pieces. For example, for the single topic "triple integrals" on the thematic map, my day-by-day schedule has double $\to$ triple integral, triple integral, practice triple integral, set up own triple integral, and change order of triple integral.

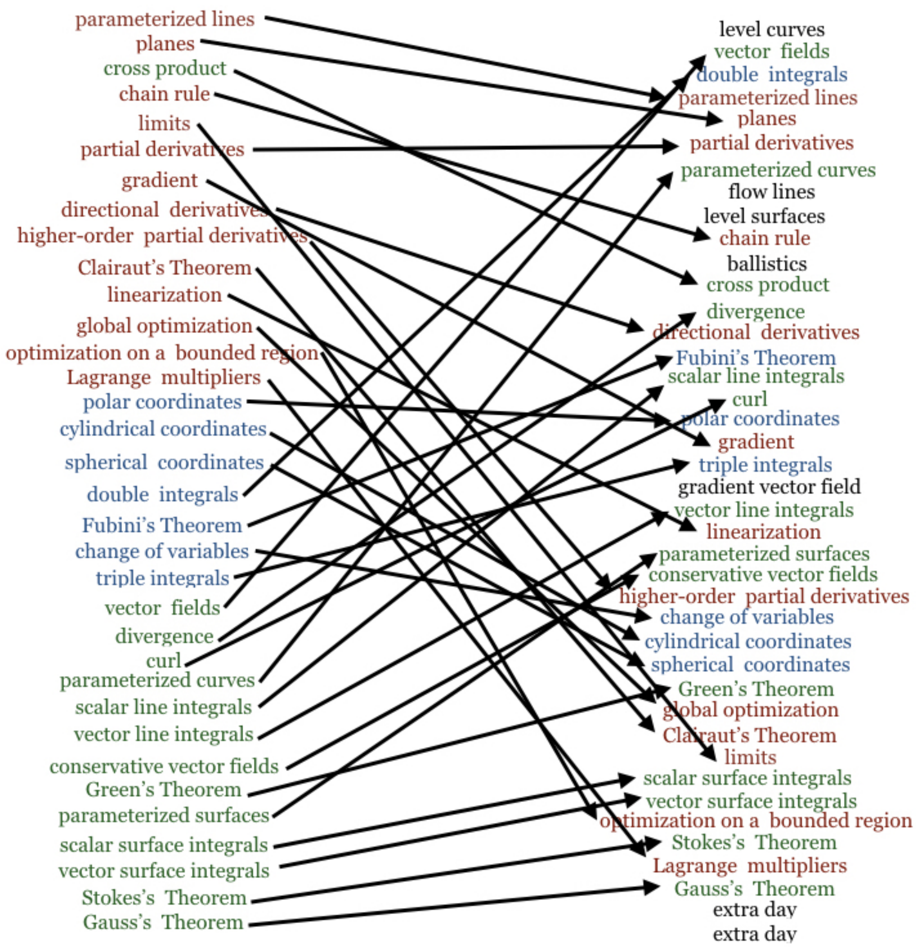


Figure 4: The lecture order (left) and the order of topic introduction in my curriculum (right). Note that derivatives, integrals, and vector calculus are segregated on the left, and integrated on the right.

Action Task 3.10. On the whiteboard, make a large calendar, where each box is a day of your class. Again, make a post-it note with each of the topics from Action Item 3.1. Place each topic on the day when it will be first introduced. Refer to your thematic map from Action Item 3.2 to make sure that students have the prerequisite tools they need for each new topic.

3.12 Arrange the problems in the book

The biggest job in constructing the book is then to decide which problems go on which days. Here are some tips.

Each problem is its own paper strip, labeled by topic with an ordering. I type out all of the problems, print them out, and cut each problem into a strip. If two problems are related and should be consecutive, I don't cut between them. Then I lay them on a large surface organized by thematic threads: all the problems on this thread in one column, all the problems on that thread in another column (see picture below). I choose a letter to associate to each thread, and then number the problems of the thread in order: A 1, 2, 3, . . . for problems about planes, B 1, 2, 3, . . . for problems about partial derivatives, and so on. This ensures that you keep the problems in the right order, and that you can pull the threads back apart later if need be.

Sometimes I am using many problems from another source, like Phillips Exeter Academy's curriculum or from a previous book of my own. In that case, before I cut the problems into strips, I

1. level curves, vector fields

2. double integrals, parametric lines, planes

3. partial derivatives, parametric curves, hyperbolic paraboloids, flow lines

4. equation of plane, level surfaces

5. chain rule, ballistics, cross product

6. change order of double integral, divergence, template for line integral

7. directional derivative, arclength, Fubini's theorem, tangent plane

8. double $\to$ triple integral, scalar line integral

9. (review for evening midterm)

10. curl, polar coordinates

11. gradient, scalar line integral, cross product as area

12. triple integral, gradient vector field, estimate vector line integral, $r\ dr\ d\theta$

13. practice triple integral, vector line integral, linearization, parameterized surfaces

14. set up triple integral, conservative vector fields, higher-order partial derivatives, intro to change of variables

15. change order of triple integral, spherical/cylindrical coordinates, meaning of higher-order derivatives, Green's theorem, template for scalar surface integral

16. extrema, surface area, more change of variables, Clairaut's theorem, intro to limits

17. second derivative test, scalar surface integral, determinant for change of variables, limits with $y = mx$

18. volume differential for polar/cylindrical/spherical coordinates, vector surface integral, limits with polar

19. (review for evening midterm)

20. introduction to optimizing on a bounded domain, spherical integral, 3D limit

21. optimizing on a bounded domain, Stokes' theorem, more 3D limits

22. Riemann sums, 3D limits with spherical, intro to Lagrange multipliers

23. proofs using Riemann sums, differentiability, Lagrange multipliers

24. Gauss's theorem

25. integrability

26. more of all that

Figure 5: A day-by-day topic introduction list for 26 days of multivariable calculus

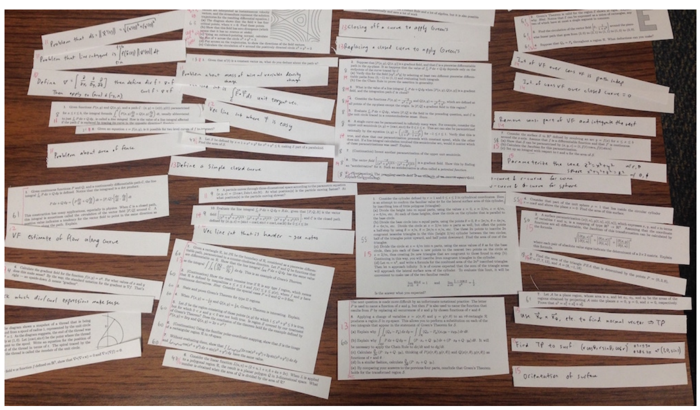


Figure 6: Each problem on its own paper strip, organized by topic threads

write the page number beside each problem, so that for example the problems from page 23 all have '23' written in the margin next to them. Since each problem is numbered on the page, this provides an ordering to the problems, and then I only need to label them by topic.

Choose which problems are in which homework assignment. I lay out one sheet of paper for each homework assignment. Now the key is to drop the strips onto the pieces of paper (not worrying about order on the page), with zero, one or two problems from each thread on each paper. I think this is the hardest part.

The order of the homework and discussion problems within each assignment: This is important, because it tends to dictate the order in which students do their homework, and the order in which problems are discussed in class. I like to use the following:

- First one or two problems: practicing something they have already learned, maybe in a new context. This eases them into their homework, and into class discussion.

- Next few problems: new big ideas for the day. They tackle this early in the homework assignment when they are full of energy, and discuss it early in class when there is plenty of time.
- Next problems: building up examples and experience with new ideas. There is work to do in homework, but these may be quick to discuss in class.
- Last problem or two: exploring new ideas that we will study more carefully later. The idea of such problems is to encourage the students to wonder about what is going on, so that they will be motivated to study the topic later. These problems are at the end, in case the student runs out of time in homework or in case they run out of time in class to discuss them, as they are not strictly necessary in order to learn the content, because students will study the same ideas in a forthcoming assignment.

Figure 7: A problem-based curriculum laid out on every available surface in my office: threads of ordered problem strips on the desk, and on the floor one page for each day of the semester

Action Task 3.11. Find an existing problem-centered curriculum, where daily assignments are delineated. Phillips Exeter Academy's curricula, available at [6], are good for looking at the mix of problems on a page, but note that a day's assignment might be just eight consecutive problems, not necessarily all on the same page. My curricula are available at [4], with one, two or three pages per assignment, as labeled.

- Identify which problems are easier and which are harder, which are explorations and which are practice, which are questions about ideas and which are computations, and so on.
- Reflect on the ordering of the problems, whether they alternate types or have several similar problems in a row.
- If looking at a curriculum where each assignment is delineated, reflect on what types of problems are at the beginning, middle and end of the assignment.

Action Task 3.12. Assemble your problem threads into a book:

- Lay out one sheet of paper for each day of your class. Write the day number at the top of each page. I also write things like "review day for exam."
- Copy each topics from Action Task 3.10 onto the top of the page for the day when you have decided to introduce it.
- Print out each of your problem threads, single-sided. I recommend two pages per sheet (half size pages) or smaller, since you will need a lot of work surface area. Consider labeling the problems so that can identify which thread each problem is in, and also its number in the thread. This way, you can pull them apart later if you need to.
- Cut each problem into its own paper strip and lay all of the strips out on your work surface.
- Drop each problem onto the page for the day when it will be assigned.
- One page at a time, decide the ordering of the problems on the page.
- On your computer, arrange the problems the way you have them on the page.

3.13 Edit your problems

It is a good idea to consider your curriculum to be a living document, that you modify each time you teach it based on previous students' experiences. This way, when a problem doesn't work out as well as you expected, you can consider it an opportunity for future improvement, rather than a failure.

Phillips Exeter Academy has a long tradition of editing their teaching materials over the summer. All of the teachers keep track of edits they want to suggest, and a committee spends several weeks changing the problems and modifying problem numbering and solutions accordingly.

Similarly, during each class, I write many notes in the margins of my copy of the problem pages. I substantially edit many problems from year to year, changing confusing wording, adding earlier problems to develop needed skills or prime students with needed ideas, or add into the book additional parts that I suggested to students in class.

The following is an example of one problem's evolution in my care. This problem develops the Fundamental Theorem of Line Integrals. Here is the original version of the problem, from Phillips Exeter Academy's multivariable calculus materials.

Phillips Exeter Academy, Mathematics 6, page 8

2. Suppose that $[P(x,y), Q(x,y)]$ is a gradient field, and that $\mathcal{C}$ is a piecewise differentiable path in the xy-plane. It so happens that the value of $\int_{\mathcal{C}} P\,dx + Q\,dy$ depends only on the endpoints of the curve traced by $\mathcal{C}$.

(a) Verify this for the field $[xy^2, x^2y]$ by selecting at least two different piecewise differentiable paths from $(0,-1)$ to $(1,1)$ and evaluating both integrals.

(b) Use the Chain Rule to prove the assertion in generality.

The same page follows up with several more problems building this idea:

3. What is the value of a line integral $\int_{\mathcal{C}} P\,dx + Q\,dy$ when $[P(x,y), Q(x,y)]$ is a gradient field, and the integration path $\mathcal{C}$ is *closed*?

4. Consider the functions $P(x,y) = \dfrac{-y}{x^2+y^2}$ and $Q(x,y) = \dfrac{x}{x^2+y^2}$, which are defined at all points of the xy-plane except the origin. Is $[P,Q]$ a gradient field in this region?

5. Evaluate $\int_{\mathcal{C}} P\,dx + Q\,dy$, where $[P,Q]$ is the field in the preceding question, and $\mathcal{C}$ is the unit circle traced in a counterclockwise sense. Hmm...

6-7. Problems about different parameterizations of a semicircle.

8. The *vector field* $\left[\dfrac{-x}{(x^2+y^2)^{3/2}}, \dfrac{-y}{(x^2+y^2)^{3/2}}\right]$ is a gradient field. Show this by finding an "antiderivative" for it. Such an antiderivative is often called a *potential function*.

When I modified this problem thread to use it the first time, I merged #2 and #8 into one problem, eliminated the others, and added hints (problem text follows below). Because there is less time in college than in high school, I had to teach the same material in fewer problems, so I added extra guidance to the problem text.

First, I added extra explanations of notation: a reminder of the definition of a gradient field, and a labeling of the vector field as **F**, to prevent the problem of students having no idea where to start. I also added the terminology *conservative vector field* and the statement of the *Fundamental Theorem of Line Integrals* to be consistent with the textbook used by the concurrent lecture-based sections of the same course. I merged the idea of #8 into the problem by defining a *potential function* and requiring the students to use it right away. Finally, since problem 2 from above is *hard* and has a lot of steps, I added the information that the Fundamental Theorem of Calculus is required for the proof, to give students a hint of how to proceed.

Math 290-3, page 26 # 1.
Suppose that $[P(x, y), Q(x, y)]$ is a *gradient field*, i.e. $[P, Q] = \nabla f$ for some function $f(x, y)$, and that $\mathcal{C}$ is a piecewise differentiable path in the xy-plane. It so happens that the value of $\int_{\mathcal{C}} P\,dx + Q\,dy$ depends *only* on the endpoints of the curve traced by $\mathcal{C}$.

(a) Verify this for the field $\mathbf{F} = [xy^2, x^2y]$ by selecting at least two different piecewise differentiable paths from $(0, -1)$ to $(1, 1)$ and evaluating both integrals.

(b) A vector field that is the gradient field for a function $f(x, y)$ is called a *conservative vector field*, and f is called its *potential function*. Find a potential function f for $\mathbf{F}$, and evaluate $f(1, 1) - f(0, -1)$.

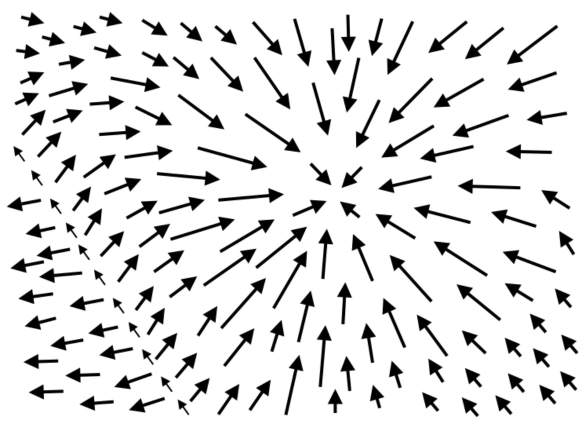
Let's call this result the *Fundamental Theorem of Line Integrals*: If $\mathbf{F}$ is a conservative vector field, and its potential function f is defined on a region containing the curve C, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\text{end point of } C) - f(\text{starting point of } C).$$

(c) Use the Chain Rule and the Fundamental Theorem of Calculus to prove this fact.

The second time I used this problem, at Swarthmore, I didn't change it. But afterwards, I decided that many students were missing the intuition behind it, perhaps because of eliminating the extra work in the original problems #3–7, especially #3. Thus, the third time I taught it, I added several new problems on the previous page to get the idea across.

In the following problem, parts **(a)** and **(b)** are just to make sure students understand what the figure depicts. Parts **(c)** and **(d)** foreshadow the part of the Fundamental Theorem of Line Integrals problem where students will compute the vector line integral of the vector field along two different paths of their choice. Part **(e)**, whose answer is "elevation change from A to B," ensures that they think about what they are doing and get the point of the problem.



Math 34, page 13 # 8.
Let $f(x, y)$ give the elevation of the point (x, y) for the region where you are hiking. The picture above shows the vector field $\nabla f(x, y)$.

(a) Where is the highest point on the map? Where is the lowest point on the map?

(b) Identify important features such as mountaintops, valleys, streams, etc., and explain how you know where they are.

(c) Mark two points A and B of your choice, not too close together. Connect A and B by a curve C_1. Estimate the value (positive? negative? big? small?) of the vector line integral

$$\int_{C_1} \nabla f \bullet d\vec{\mathbf{s}}.$$

(d) Connect A and B by a different curve C_2. Make C_2 very different from C_1. Estimate

$$\int_{C_2} \nabla f \bullet d\vec{\mathbf{s}}.$$

(e) What is the physical meaning of the integral $\int_C \nabla f \bullet d\vec{\mathbf{s}}$ in this context?

Students discussed the above problem one day, and had the following problem for homework that same night, so that the ideas would be fresh in their minds. Then the next day, they did the original problem, with **(a)**, **(b)** and **(c)** (now called **(d)**) the same, and I added in an additional part **(c)**, to make them connect this idea to the mountain / gradient / hiking problem on the previous page:

> *Math 34, page 14 # 3.*
> Suppose that $[P(x, y), Q(x, y)]$ is a *gradient field*, i.e. $[P, Q] = \nabla f$ for some function $f(x, y)$, and that $\mathcal{C}$ is a piecewise differentiable path in the xy-plane. It so happens that the value of $\int_{\mathcal{C}} P\,dx + Q\,dy$ depends *only* on the endpoints of the curve traced by $\mathcal{C}$.
>
> **(a)** Verify this for the field $\mathbf{F} = [xy^2, x^2y]$ by selecting at least two different piecewise differentiable paths from $(0, -1)$ to $(1, 1)$ and evaluating both integrals.
>
> **(b)** A vector field that is the gradient field for a function $f(x, y)$ is called a *conservative vector field*, and f is called its *potential function.* Find a potential function f for $\mathbf{F}$, and evaluate $f(1, 1) - f(0, -1)$.
>
> Let's call this result the *Fundamental Theorem of Line Integrals*: If $\mathbf{F}$ is a conservative vector field, and its potential function f is defined on a region containing the curve C, then
>
> $$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\text{end point of } C) - f(\text{starting point of } C).$$
>
> **(c)** Give a geometric explanation of why this is true. *Hint*: Recall Page 13 # 8.
>
> **(d)** Use the Chain Rule and the Fundamental Theorem of Calculus to prove this fact.

Action Task 3.13. As you teach your class, keep your own copy of your text, and make lots of notes in the margins and on the problem text for how to improve them for the next time you teach.

4 Creating a discussion-based classroom

4.1 Classroom methods

In all the methods described below, there is a problem-based curriculum where the students learn and discover the material through doing problems, and class time is spent discussing solutions to the problems. Here are several ways to organize class time, depending on the size of the class and the number of instructors.

4.1.1 One group, one instructor

Classical Phillips Exeter Academy style.

- 8-13 students, one instructor
- whole-class discussion facilitated by instructor
- student presentations of solutions to class
- students and instructor seated at oval table
- One or two problems a week possibly written up and handed in

When students enter the room, they choose a problem from the previous night's homework. The first five minutes of class is spent writing their solution on the board, alone or with a classmate. For the remainder of the period, students present their solutions, standing at the board next to their work and explaining their methods. Other students and the instructor ask questions, point out errors, make connections to other problems, and so on.

Well suited to 50-minute classes. A whole-class discussion for 75 minutes is too long; better to spend the last 20 minutes with students in partners at the board working on new problems.

Pros: The instructor is present for all of the discussion that occurs in the room, and allowing them to ensure that solutions are correct, and to balance and direct the class discussion. The instructor hears all questions and comments of all students, ensuring that the instructor has a sense of where all students are in their level of understanding at all times. Shy students have a structured opportunity to participate, by presenting their solution at the board, which they can practice in advance.

Cons: Shy students can be uncomfortable speaking spontaneously in front of a group, and may need coaching and encouragement to participate in class when not presenting at the board. Students may be hesitant to ask questions or point out errors in the work on the board, as they feel it is rude to their friend who is presenting.

4.1.2 Several groups, several instructors

- 12 or more students, instructor and TAs
- students divided into two or more groups
- student presentations to group and then instructor/TA
- students standing at the board
- homework is not handed in, except perhaps one or two problems a week

When students enter the room, they are assigned to a group, with each group having 6-9 students, and the number of groups equalling the total number of instructors and TAs.

First five minutes of class: Each group functions as in the previous method, with each student choosing a homework problem and writing their solution to it on the board.

The remainder of the class is the discussion, which has two parts. The *second part of the discussion* is that, when the group is ready, the instructor or TA joins them, and chooses a student to explain each problem, who is not the student who wrote it up. The *first part of the discussion* is that, in preparation for this, students explain their solutions to each other, ask each other questions, and point out errors, entirely or almost entirely without input from the instructor or TA.

Pros: Students are highly motivated to ask each other questions in the first part, to avoid looking silly in front of the instructor later. Physically standing at the board together promotes student engagement. The primary instructor hears from each student on average every other day, so they have a sense of how well each individual understands the material. Every explanation from every student is heard by some instructor, who can provide feedback and guidance.

Instructors can choose the student who presents each problem to achieve some goal: for example, if a student has expressed anxiety about their understanding of some topic, I choose them to explain the problem on that topic, and they invariably learn that they can do it. I generally choose the self-identified weaker students to explain harder problems, and the self-identified stronger students to explain easier problems.

Cons: Requires having one or more TAs in the room. Ideal for a 75-minute class period; the three phases of the class may not fit well into a 50-minute period.

4.1.3 Many groups, one instructor

- 20-30 students, one instructor
- small-group discussions mainly independent of instructor
- students discuss problems and write on board spontaneously
- students seated at tables or in a semi-circle of desks against a chalkboard
- all homework problems are carefully written up and handed in on a later day

When students enter the room, they are assigned to a group, with each group having 4-6 students. For the entire class period, students discuss their solutions to homework problems, calling over the instructor when desired but otherwise operating independently. Instructor checks every day to make sure every student has attempted every problem before class. After class, students must write up their solutions and hand them in, to ensure that they understood everything correctly.

Pros: Places the responsibility of learning directly on the student. Usually results in energetic and thoughtful discussions in every group for the full period. Every student in the group generally participates throughout the period. Students are more willing to talk in small groups, and it is easier for them to talk freely to their peers, without the instructor present.

Cons: There is danger that students will come to consensus on a wrong answer and move on, since no one is checking their work; this is the purpose of requiring that students write up and hand in solutions. The instructor only hears the questions and comments of students occasionally, and otherwise does not have a sense of where individual students are in their understanding.

4.2 Incentives

For each thing you want your students to do, you should create an incentive for them to do it.

Things I want students to do: attend class, learn each other's names, think hard about every homework problem and try to solve every one before class, participate actively in class, ask questions when they have them, answer other students' questions, point out mistakes, suggest alternative solution methods, understand the course material.

Incentives I use to make these things happen: peer pressure and grades.

One semester, I had a class with 17 students, in a full-class discussion (as in §4.1.1), where they just would not talk, no matter what I tried. So halfway through the semester, I broke them into two groups and told each group that they had to find the correct answers and understand all of the solutions, by the end of class. I told them that I would come around and for each problem, I would ask a student who had *not* written that solution on the board to explain it.

To my surprise, standing by the board in groups of just 8 or 9 students, they spontaneously did everything I had been pleading for them to do all semester: explained things to each other, asked each other questions, and pointed out mistakes. My act of frustration turned into the "several groups, several instructors" method described in §4.1.2. The students didn't want to look silly in front of me or their peers, which was enough incentive to get them to engage with the problems. In fact, I had brought small prizes, which I gave out to each student who correctly explained a problem, but after two classes it was clear that peer pressure was a powerful enough incentive on its own.

When I have students in multiple groups, especially in the "many groups, one instructor" method explained in §4.1.3, I need to incentivize students to ask questions and not just write down garbage for the solution in their notebook, and for this I rely on grades. For this method I require that students hand in solutions to each assignment, one or two classes after they have discussed it in class. Students are highly motivated by grades, and under this method they actively engage in class.

I have had problems, especially after the middle of the semester, with students not doing their homework before class, and relying on classmates to explain all of the problems to them. This is rude (sadly, peer pressure isn't enough to prevent this) and a waste of class time. So now I check their homework: I have every student point out to me where they worked on every problem that was assigned. When there is a blank space in their notebook, I start in on them with a paragraph of why it is important for them to do their homework before class. It doesn't take much of this before they are thoroughly chastened and decide to work on all of the problems for next time.

desired behavior	**incentive**
students do homework before class	instructor checks notebook for attempts
students ask each other questions in class	students must turn in solutions in for a grade
students answer each other's questions	peer pressure

If you are having trouble getting your students to engage in some desired behavior, I encourage you to resist the temptation to *tell* them what you want them to do, and instead set up *incentives* so that they will adopt the desired behaviors on their own.

4.3 The first day of class

The first day sets the classroom atmosphere for the entire term. You should prepare very thoroughly for the first class, and make sure that each thing you do, and that you ask the students to do, has a purpose. Because it is so important, I include here everything I do on the first day. One purpose of this is to give you a framework to start with. The other purpose is to drive home the point that *everything you do on the first day is extremely important.*

Here's what I do on the first day of class.

- I ask the students to arrange the classroom chairs in a circle big enough to include everyone. *Purpose: to show that this is a discussion-based course and that we are all equal.*

- There are two pages for students to pick up: the syllabus, and the first page of homework. *Purpose: students have the information they will need for the course and for the next day, and they get in the habit of picking something up from the table each day.*

- At the start of class, I very briefly explain how the class will run: there will be no lectures, you will have challenging problems for homework, and we will spend class time with you explaining your solutions to each other and asking each other questions. *Purpose: to make sure everyone knows the setup and expectations for the course.*

- I tell the students that we're going to go around the circle, and each student will read a paragraph of the syllabus. We do this. *Purpose: to hear everyone's voice, to ensure that all students have read the syllabus, and to construct the figure that comes next.*

- I go to the board and draw the following three pictures. Each dot symbolizes a person around the circle. The first picture is a tangible example, the graphical representation of the "discussion" that we just had in reading the syllabus. For the second example, I explain what the Socratic method is (and explain that the picture is the same for the "lecture with student input" method), where all student responses go through the teacher, and explain that our discussions will *not* be "shaped like a fan" like this. The third picture sets the goal for our class discussions, that they will be "shaped like a web."

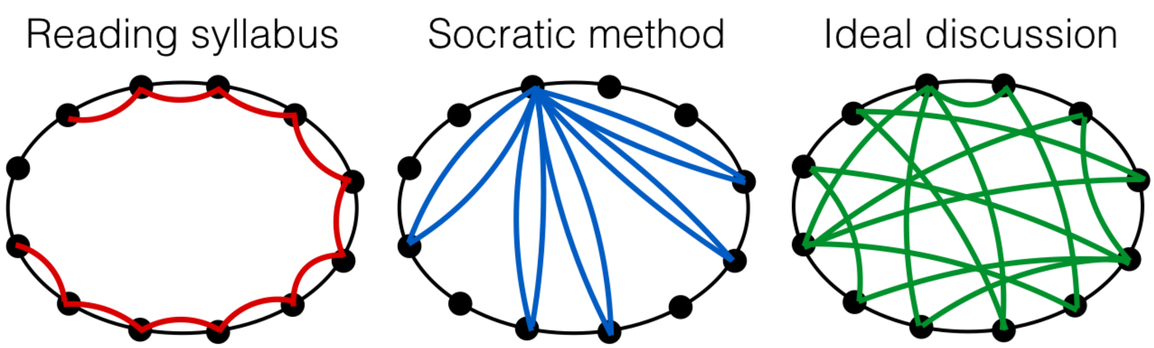


Figure 8: Three types of class discussions

- I tell students: "I assume that none of you have ever been in a math class like this before, so you don't have any idea how to do it, and maybe you're a little worried about it. That's okay – I'll teach you how to do it, how to present your solution to your classmates, and how to ask questions. At the beginning, I'll give you a lot of help, and remind you about what to do, and as you get better at it I'll help you less." *Purpose: to reassure them that I know that this is an unfamiliar challenge for them, and to tell them my role in the classroom.*

- I tell the students that we are going to go around the circle and each person will say their name, a story about why they were given that name or a way that we can remember their name, and also what they are excited about in this class. We do this. *Purpose: to make sure that each student has practice speaking to the whole class, to help them learn each other's names right away, and to create an expectation that we are all excited to be here.*

- I ask them to take out the homework sheet they picked up. One side is the problems for today in class, and the other side is their homework problems. I ask them to form a group of two or three, go to the board, and work on the problems for today. *Purpose: the problems for the first day are designed to be exploratory, so that the students play around with examples, make conjectures, and get excited about the ideas that we will be studying over the course of the term. Also, they are forced to practice talking with other students about math.*

  For classes with presentations:

- With about five minutes left in the period, I stand in the middle of the room and ask everyone to stop working. Then I ask for a volunteer group to explain their solution to the first problem. "Please explain what the problem was asking, and explain what you did, and point to your work on the board," I tell them. They do this. If I'm lucky, another student will ask them a question, and they'll have to explain something. Otherwise, I tell the class that this is a great time to ask a question, and usually a student asks one. Then I repeat this for the second problem. *Purpose: to have students practice standing at the board and explaining their work, in a situation where everyone in the room has thought about the problem.*

- To close the class, I explain what will happen in the second class: "When you come in to class, I will have written the problem numbers on the board − 1, 2, 3, and so on. Don't sit down, just put down your bag and then choose a problem and write up your solution on the board. Make sure to write your whole solution, probably including a picture, not just your answer! After about five minutes, you'll be done, and we'll all sit down, and then I'll ask whoever put up number 1 to explain it. When it's your turn to explain your problem, stand up next to it, explain what the problem was asking, and then explain what you did. When you're in the circle, your job is to pay attention to the person at the board, and then ask questions whenever you have them. The goal is that we will discuss each problem until everyone understands what's going on, and then we'll move on to the next one." *Purpose: to let the students know what to do when they enter the classroom the next time, so that they will be ready and prepared.*

  For classes without presentations:

- I give a 10-minute speech explaining my educational philosophy, and why the course is organized as it is, which appears in modified form as the introduction to this book. *Purpose: to get students excited about the course structure, and to emphasize that their struggle is an essential part of the course, in order to increase student buy-in.*

  For all classes:

- I thank the students for coming, and class is over. *Purpose: to acknowledge their efforts, and to have a clear endpoint to the class.*

4.4 The second day of class, and all following days

The first day is absolutely crucial for setting the atmosphere of the room. The second day is also crucial, for setting the tone of how each class will run. It is key to establish a culture that is both friendly and inquisitive, so that students are kind to each other and feel comfortable asking each other questions, and also so that students are ready to work, struggle, and learn.

Below is what I do, after a few years of experience. Everyone who teaches in this method runs their class differently, and some do things very differently. I encourage you to try this way, keep what you like, and modify it to make it your own.

Arrangement of the room: It is ideal to have an oval table, with chairs around it, and blackboards on every wall. If you don't have this, choose the part of the room with the most board space (it's ideal if it's a corner with boards on both adjacent walls), and arrange your desk-chair combinations into an oval in that corner. Leave about two feet between the backs of the chairs and the walls, to allow space for students to stand at the board. Leave some space between desks so that everyone has enough personal space, and also so that it's easy to get up and go up to the board.

Setup for the class: Write the problem numbers, for example #1 − #6, partitioned out along the top of the board, with a large enough space for each solution. If you have a lot of students and your problems have multiple parts, consider writing up each part as a separate problem (e.g. 1a, 1b, 1c) so that more students need to put up their work. Place a table near the room entrance with anything they need to pick up, such as the next day's problem set or any materials (string, transparencies, etc.) that they may need.

When students walk in: Greet them by name, and ask them how they are doing. Allow them to drop their backpack in a chair, but not to sit down − "sitting is a special prize for people who have already put up a problem!" I tell them − and remind them to get out their homework and choose one of their solutions to write up on the board. If someone else is already doing the problem they wanted, they are welcome to work together.

If only the "hard problems" are left, and the students arriving later claim that they did not solve any of those problems, ask them to put up what they have anyway so that the class has somewhere to begin, and praise their courage for being willing to do so.

Five minutes into the class period: Most students should be done writing up their solutions, and sitting down. Stand up and announce loudly, "Okay, it's time to start − those of you at the board, please finish up what you're writing and sit down, and you can finish when you get up to the board." Make any announcements, and ask if students have any announcements or things they'd like to invite their classmates to. Then sit down and say, "everyone please open your books to page n, and open your homework to where you did the problems, and let's talk about #1."

The notion of my standing up and announcing things loudly − at the beginning of class, at the end of class, and anytime during the class when I decide to say something − is a recent innovation in my teaching. Students are naturally hyper-aware of what the teacher is doing. If I get in the habit of doing subtle things, talking quietly, saying things that students may miss if they are not paying close attention to me, then they will learn to pay *very* close attention to me: my breathing, my posture, my facial expressions. That is not what I want; I want them to ignore me and pay attention to each other. So, when I decide to say something, I say it decisively, in a way that no one will miss, with the perhaps counterintuitive goal that this will make them notice me *less*.

How to present a problem: The student (or group of students) should stand up at the board, next to where they wrote up their solution. They should stand to the side of it so that they are not blocking anyone's view, and face the class. First, they should explain what the problem was asking, and then explain what they did to solve it, pointing to what they have written on the board when they are talking about that part.

Then, students ask questions: If you are lucky, students will ask questions on their own. The student asking the question (say, Morgan) and the student at the board (say, Alex) will both look at you. Look at Morgan and point to Alex. "Ask Alex!" Alex will answer Morgan, and then will look to you for reinforcement.

This is a great time to talk about eye contact: remind them that we want class discussions to look like a web, with students talking with each other and not to the teacher, so they should look at each other when asking and answering questions, and not at you. I tell students, "you may find that when you are talking, I am not looking at you. That's not because I'm being rude; it's because I'm forcing you to look at other students instead of making eye contact with me."

Remind students that the problems are hard, and that they are designed to make students get stuck, and this is the time to ask questions. "Don't just copy down the work from the board and think that you'll figure it out later!" I tell them. "Ask Alex a question right now, while we are all here to help you." If a student (say, Dale) reluctantly asks a question, allow Alex to answer it and for them to have the discussion, and then praise Dale: "great question! I'm so glad you asked that. Were any others of you wondering the same thing?" (You raise your hand. Hopefully a few students raise their hand, too.) "Yeah, I thought so! Dale, thank you for being willing to take one for the team and ask about that."

When discussion of that problem is over, say, "let's talk about problem 2," and repeat the above, and so on for the rest of the problems.

If there is time left over: Here are a few options.

- If there are less than two minutes left, let them out early.
- If there are three or four minutes left, go to the board and explain something in more detail: take something that the class discussed that day, and tell them something interesting about it that they would have no way of finding out about, or something that relates to your life.
- If there are five to eight minutes left: say, "Get into groups of two or three, go to the board, erase what you find, and work on problem 3 of tonight's homework. Ready... GO!" Assign them to work on the problem that is most conceptually difficult, so that they will start thinking about it as early as possible.
- If there is quite a bit of time left, make up a challenging problem that introduces them to a conceptually difficult idea that is coming up, maybe two homeworks in the future. Explain the problem, write it down precisely on the board, draw a box around it, ask if anyone has questions about what you're asking them to do, and then say as above, "Get into groups of two or three, go to the board, erase what you find, and work on this problem. Ready... GO!"

The purpose of saying "ready... GO!" is that if your explanation just sort of ends, then your students will just sort of sit there, and precious seconds will bleed away while they wait for each other to make the first move towards standing up.

The end of class: Whether students are sitting in a circle talking about homework, or at the board in groups working on new problems, it's important to have a clearly defined end to the class period. I like to stand up and say, "Okay, great job today! Remember to turn in your homework and pick up the next one. If you have time, please erase the boards. See you!"

5 Assessment

The previous two sections described how to create a problem-centered, discussion-based course. My practice has been to keep every other aspect of the course, in particular exams and grading, the same as in a "traditional" course. I do this to make students feel as secure as possible with the structure of the course, by making it familiar. In a department where problem-centered, discussion bases courses are the standard of instruction, using mastery-based learning or assessment rubrics could work well. I think it is important to make students feel as comfortable as possible in their first experience with student-centered classroom environment, and nearly all of my students are having their first such experience with me, so I stick to the basics.

My course grades are based on class participation (15%), graded homework (15%), first midterm (20%), second midterm (20%), and final exam (30%). This is in line with other courses in the math departments where I have worked, the only difference being that the total of 30% that I devote to participation and homework may also include things like online homework and attendance.

5.1 Class participation

From my syllabus:

> In class, you will work on the homework problems in small groups. Your active participation is crucial! The purpose of class time is (1) to get un-stuck on homework problems where you were stuck, and (2) to explain the problems you understood to other students who were stuck. I expect you to contribute ideas whenever you have them, while allowing for a good balance and encouraging all students to contribute their voices. We will meet to discuss your work in this class and come to agreement on your participation grade, between 1 and 15.

The purpose of including participation in the course grade is to ensure that students make good use of the time we spend together: show up, ask questions to help their understanding, and answer other students' questions to help others understand. The purpose of having students choose their own grade is to force them to think critically about whether they are making the most of the opportunity presented to them in class.

I meet with each student for 10 minutes, in the week or two after the first midterm. I send them an email in advance telling them that must come in with a participation grade between 1 and 15, and I remind them of the guidelines, from the syllabus quoted above, for assessing their work towards determining this number. In our meeting, I ask them the following questions:

1. What are some things that you enjoy about the course, that we are doing well?
2. Are there any things that you dislike about the course, that we could change to make things better?
3. Do you have a number, from 1 to 15, for your participation?

   Students usually suggest a number between 12 and 14.5. So I ask a follow-up question:
4. Okay, it sounds like you're doing a lot of things well in class. That's great! Of course, you'd like to improve your participation to a 15. What are some goals you have for yourself for the second half of the course, in order to get as much as possible out of class time?

   I write down their goals in my notebook. Some common themes are "I want to try to explain problems that I'm not sure about" and "I want to ask more questions." I ask a follow-up question:
5. Nice, that's a great goal. Okay, we have class again tomorrow. Can you state a concrete action you are going to take in class tomorrow, to work towards achieving this goal?

If time permits at the end of the semester, I have a second (optional) meeting with each student, where I follow up and ask them about their specific goal, how they worked towards achieving it and how that went. I also allow them to increase their participation grade accordingly. Students often take their goal very seriously, and tell me in the end-of-semester meeting how they asked one question per week, for example, and worked up to one question per day, and then multiple questions per day. It's a powerful exercise for those who embrace the opportunity.

5.2 Graded homework

From my syllabus:

> Daily homework is the core of this class. For each class meeting, you will do a set of challenging problems. I expect you to try hard to solve every problem, and spend three to four hours on each night's homework. Your goal should be, for each problem, to either solve it *or* to get to the point that you are really stuck. I encourage you to work together with other students. It is essential that you give each assigned problem a good effort, and solve it if possible, *before* class. In your notebook, you must write down the relevant information for the problem, draw a picture, and record your efforts toward a solution, for *every* homework problem. Every Thursday, you will hand in the previous week's homework (usually the Thursday and Tuesday assignments). All of it will be graded for completeness, and several problems will be carefully graded.

For the proper functioning of the in-class discussions, students must have worked on all of the homework problems. To make this happen (see §4.2 on incentives) it is important to do a notebook check every day. For the first five semesters I taught in this method, I did no notebook checks. When I started doing them in my sixth semester, the effect was that nearly every student who attended class had attempted every problem before class. This makes the discussions go better for each group, and also makes class time more useful for each student. I don't record anything; if a student didn't do the homework, I simply ask if the student's life is going okay, explain the purpose of doing homework before class, and encourage them to get some sleep and start earlier next time.

I am fortunate to have undergraduate graders. As explained in the quoted syllabus above, students hand in two assignments at a time, once a week, and the grader takes them. With 4 grader hours per week for 30 students handing in two assignments each with roughly 8 problems, I have the grader grade one problem per assignment "carefully" (check everything, write comments, and grade it out of roughly 10 points), and grade the rest mostly for existence (not correctness) on a scale of 2 points each.

5.3 Exams

I give "normal exams" (with two midterms and a final). Several times, I have even taught a discussion-based course parallel with lecture-based sections of the same course, using common midterm and final exams (see [3]), in which case my exams were identical to exams for lecture-based courses. Most of my exam questions are of the form "compute this…", just like everyone else.

I am not philosophically opposed to projects, partner or group tests, or exploratory problems on exams, but I want students to feel as secure as possible in the course, so I want exams, which always cause stress, to be as predictable as possible. The problem-centered course format with exploratory and discovery problems causes students to learn really well, for about 30 hours in class over the course of the semester, and two or three times that outside of class. This learning is the most important thing for me. They also spend a couple of hours exhibiting that learning through traditional exams, which work fine and keep added anxiety to a minimum.

I grade exams out of 100 points, write comments on them, return them to the students, and don't accept any kind of test corrections or other ways of earning back points. If a lot of students do poorly on a certain problem, I may add a few problems on that subject to upcoming homework assignments, to correct students' understanding and allow for practice. I will also put a very similar problem on the final exam, and tell them that I am doing so, in order for students (who are usually frustrated about having done it incorrectly when they felt like the understood it) to have a second chance to demonstrate their mastery of that topic.

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[3] Diana Davis, *Inquiry-based learning in a first-year honors course*, PRIMUS, 28(5), 387-408 (2018), available at `https://arxiv.org/abs/1606.08834`

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